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Stability of slowly diverging jet flow

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Coherent axisymmetric structures in a turbulent jet are modelled as linear instability modes of the mean velocity profile, regarded as the profile of a fictitious laminar inviscid flow. The usual multiple-scales expansion method is used in conjunction with a family of profiles consistent with similarity laws for the initial mixing region and approximating the profiles measured by Crow & Champagne (1971), Moore (1977) and other investigators, to deal with the effects of flow divergence. The downstream growth and approach to peak amplitude of axisymmetric wave modes with prescribed real frequency is calculated numerically, and comparisons are made with various sets of experimental data. Excellent agreement is found with the wavelength measurements of Crow & Champagne. Quantities such as the amplitude gain which depend on cumulative effects are less well predicted, though the agreement is still quite tolerable in view of the facts that this simple linear model of slowly diverging flow is being applied far outside its range of strict validity and that many of the published measurements are significantly contaminated by nonlinear effects. The predictions show that substantial variations are to be expected in such quantities as the phase speed and growth rate, according to the flow signal (velocity, pressure, etc.) measured, and that these variations depend not only on the axial measurement location but also on the cross-stream position. Trends of this kind help to explain differences in, for example, the preferred Strouhal number found by investigators using hot wires or pressure probes on the centre-line, in the mixing layer or in the near field.

1. Introduction

Turbulence research has advanced rapidly in the last decade with the widespread recognition of orderly large-scale structure in many kinds of turbulent shear flow. Largely because of the jet-noise problem, turbulent jets at Reynolds numbers of 10^4 and beyond have been subjected to the most intensive study, and some measure of agreement seems to have been reached among investigators on the general properties of the coherent motions. A number of papers deserve specific mention in any brief description of previous work. Mollo-Christensen (1967) measured the near-field pressure fluctuations, and found that they come in quite well defined packets, the packets having a similar structure and being random only with respect to the place and time of their origin. Ronneberger (1967) used a loudspeaker in the jet-pipe to force a jet to undergo a controlled uniform periodic fluctuation at the nozzle. He measured the amplitude of the backscattered pressure signal in the pipe at various flow speeds and frequencies, and found that the reflexion coefficient rose above unity in some circumstances, reaching a peak at combinations of the Mach number U_0/a_0 and Helmholtz number $\omega D/a_0$ which give the Strouhal number $St = \omega D/2\pi U_0$ a value around 0·3. At this condition, particularly intense vortex shedding took place from the nozzle. Ronneberger did not, however, measure the field outside the jet-pipe. That was done in the well-known work of Crow & Champagne (1971), who measured the turbulent velocity field by hot-wire anemometry, and by Crow (1972), who measured the distant noise field. Crow & Champagne's approach, in which the latent structure is deliberately raised above the random background by coherent forcing, has been recently followed by Chan (1974*a*–*c*), who measured pressure fluctuations at various axial and radial locations. Other workers have preferred to study the orderly structure without external forcing, relying on either narrow-band cross-correlations (Lau, Fuchs & Fisher 1972; Fuchs 1972) or 'eduction' techniques to filter out the quasi-deterministic motions. Much recent experimental and theoretical work on orderly structure in a variety of turbulent flows was described by the contributors to the 1974 Southampton Colloquium on Coherent Structures, and summarized by Davies & Yule (1975).

Crow & Champagne showed that a 'small' level (typically 1 % in their experiments) of coherent periodic forcing at the exit plane excites axisymmetric wave modes which 'phase-lock' the initial six diameters or so of the jet, and bind almost all of the turbulent energy into the purely periodic motion. For a given forcing level at the exit there is a 'preferred mode' which suffers the greatest total amplification (by a factor of about 18 on the centre-line axial velocity field), and in the Crow–Champagne experiments that mode was shown to have Strouhal number 0·3, phase speed $0{\cdot}71U_0$ and wavelength $\lambda = 2{\cdot}38D$, reaching its peak level at $5{\cdot}5D$ from the nozzle. These figures were all taken from measurements of the axial fluctuation velocity on the centre-line. If the exit-plane forcing level exceeds 1 % or so, the fundamental mode behaviour is strongly controlled by nonlinearity, though only a single (second) harmonic is measurably excited. The nonlinearity does, however, introduce a saturation effect whereby the fundamental velocity amplitude can never exceed 20 % of the exit velocity U_0 anywhere, whatever the forcing amplitude at the nozzle.

Other workers have found somewhat different values for these quantities; for example, Fuchs (1972) gives the preferred Strouhal number as 0·5, while Lau *et al.* (1972) find it to be 0·6. These discrepancies are to some extent cleared up by the work of Chan (1974*a*). His values based on centre-line pressure fluctuations agree well with those of Crow & Champagne, based on centre-line velocity fluctuations, but values derived from pressure fluctuations in the mixing region and the near field show substantial deviations from each other and from the centre-line values. This is an important point – that growth rates and wavelengths depend significantly on the *radial*, as well as the axial, location at which they are measured – and we shall return to it later.

Another cause of discrepancies lies in the fact that nonlinear effects are actually significant at much *lower* forcing levels than were used by Crow & Champagne. Moore (1977) has carried out a comprehensive study of jet response to forcing, and has found that nonlinear effects are detectable at forcing levels as low as 0·02 % of the exit velocity. This means that the factor of 18 quoted by Crow & Champagne for the peak amplification of the preferred mode must be an underestimate, suffering from the saturation effect, and a peak amplification in excess of this value would be expected at really small forcing levels. In fact Moore found that, for the centre-line axial velocity fluctuation, the amplification at Strouhal numbers around 0·5 rises from a factor around 18 to a factor above 60 as the forcing level falls from the Crow & Champagne value of about 1% to less than one tenth of this value. These figures relate to the narrow-band filtered signals. Further evidence of the presence of significant nonlinearity comes from measurements by Moore (1977) and Bechert & Pfizenmaier (1975) showing an increase in the *broad-band signal* (of both the pressure and the velocity fluctuations within the jet *and* of the far-field sound pressure) at exit forcing levels greater than, say, 0·1 %. In both those sets of experiments, several harmonics of the fundamental forcing tone were still measurable above the broad-band noise, with levels decreasing by about 15 dB from one harmonic to the next (so that as far as harmonic generation is concerned the process seems to be only *weakly* nonlinear).

Turning to a theoretical framework for these results it is natural to attempt an explanation in terms of the instability modes of the mean velocity profile, on the basis that the turbulence establishes an equivalent laminar flow profile as far as large-scale modes are concerned. (Fine-scale mixing-layer turbulence may also act as an eddy viscosity (Crow 1968) though that is an aspect yet to be treated properly, and we ignore it here.) Crow & Champagne laid stress on nonlinear instability mechanisms, describing qualitatively how a preferred mode might arise from a combination of nonlinearity and dispersion for the modes on a 'top-hat' jet. While there is no doubt that nonlinear mechanisms are dominant at exit-plane forcing levels of 1 % and above, we restrict ourselves to much lower forcing levels, which in any case are probably more representative of jet engine tailpipe conditions, and argue that a reasonable description of the wave modes should be possible on linear theory. That view is supported by the work of Michalke (1971), who took the mean velocity profile measured by Crow & Champagne two diameters from the nozzle, worked out the phase speed and amplification rates for spatially growing waves for this profile (as if it were the profile of a strictly parallel inviscid flow), and showed that the results agreed well with the Crow & Champagne measurements around $x = 2D$. In particular, Michalke showed that the finite momentum thickness of the profile leads to the existence of a most rapidly amplified mode which at $x = 2D$ has a Strouhal number close to 0·3. The same conclusion is reached in the recent theoretical and experimental study by Mattingly & Chang (1974).

Quasi-parallel flow theory is, however, unable to properly predict variation of a mode characteristic with radial position, nor, for that matter, with axial position, for it will be shown in § 2 that the characteristics of a mean flow slowly changing in the axial direction are not identical with those of the parallel flow coincident with the local flow at each station. Thus the streamwise location at which the peak signal occurs does not in general coincide with the location at which the locally parallel flow sustains a neutral wave (at any particular Strouhal number).

An obvious step to take to overcome these difficulties is to incorporate the effect of axial variation of the mean flow into the analysis. A number of workers (e.g. Liu 1974; Morris 1971; Ko, Kubota & Lees 1970; Chan 1974*b*) have done this using integral formulations involving, essentially, an energy equation with turbulent dissipation, and a 'shape assumption' on the mode form, to get an equation for the amplitude variation with x. A (theoretically) more satisfactory method for slowly varying flows has recently been given by several authors. Bouthier (1972) and Gaster (1974) have examined the laminar boundary layer, Weissman & Eagles (1975) the flow in a slowly diverging channel and Karamcheti (1973, private communication) the plane shear layer of tanh-profile with thickness proportional to x. These treatments all involve a WKB or 'slowly varying' type of approximation which is readily formalized by a 'multiple-scales' argument.

The purpose of this paper is to apply the 'slowly varying' method to the evolution of axisymmetric disturbances on the initial part of a circular jet, using a form for the mean velocity which reasonably approximates the Crow & Champagne type of family of profiles. We aim to show how theory and experiment are in good general agreement as to the variation of phase speed and amplification rate with Strouhal number and axial and radial location, and as to the total amplification of the wave modes and the axial positions at which the peak amplitudes occur. We ignore interaction between the forced wave modes and mixing-layer turbulence. The conditions for the validity of Crow's (1968) theory of the viscoelastic resistance offered by fine-scale turbulence to weak large-scale straining seem to be met by this interaction, but it is not a straightforward matter to apply that theory in the present context, and neither is it yet clear that the attempt would be worthwhile. The problem is then just one of the instability of a slowly varying fictitious inviscid flow. This may imply that only the *growth* of waves can be followed; for decaying modes there may not exist any continuous solution to the inviscid equations of parallel flow (Betchov & Criminale 1967, p. 80), and that might be true for diverging flow as well.

Finally, we confine attention to modes with prescribed *purely real frequency*. In the parallel flow context these modes would suffer exponential spatial growth, a fact which, coupled with the elliptic nature of the differential equations, has led many workers to doubt the self-consistency of spatial instability theory. If allowance is made for the spread of the velocity profile, however, these doubts seem to be unfounded. The region of apparent spatial growth is bounded, and the exponentially diverging mode can be regarded as merely an inner solution which can be matched to an outer solution which attains a bounded amplitude everywhere consistent with linearity, or so, at any rate, it appears in our problem.

2. Stability analysis for diverging jet flow

Take cylindrical co-ordinates (x, r, θ) in which the mean stream function is $\overline{\psi}(x,r)$ and the stream function of axisymmetric disturbances is $\psi'(x,r,t)$. $\overline{\psi}(x,r)$ is regarded as prescribed, for example from measurements or from some similarity hypothesis about the mean jet flow. The linearized equation for ψ' is

$$\frac{\partial}{\partial t}D^2\psi' + \frac{1}{r}\left(\frac{\partial\overline{\psi}}{\partial r}\frac{\partial}{\partial x}D^2\psi' + \frac{\partial\psi'}{\partial r}\frac{\partial}{\partial x}D^2\overline{\psi} - \frac{\partial\overline{\psi}}{\partial x}\frac{\partial}{\partial r}D^2\psi' - \frac{\partial\psi'}{\partial x}\frac{\partial}{\partial r}D^2\overline{\psi}\right) + \frac{2}{r^2}\left(\frac{\partial\overline{\psi}}{\partial x}D^2\psi' + \frac{\partial\psi'}{\partial x}D^2\overline{\psi}\right) = 0, \quad (2.1)$$

with the definitions $u_x = r^{-1}\partial\psi/\partial r$ and $u_r = -r^{-1}\partial\psi/\partial x$ for the velocity components, and the notation

$$D^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}.$$

In the usual parallel flow approximation, (2.1) reduces to

$$\left(\frac{\partial}{\partial t} + \overline{U}(r)\frac{\partial}{\partial x}\right)D^2\psi' - (\mathscr{D}^2\overline{U})\frac{\partial\psi'}{\partial x} = 0 \quad (2.2)$$

with $\overline{U}(r) = r^{-1}\partial\overline{\psi}(r)/\partial r$ and $\mathscr{D}^2 \equiv \partial^2/\partial r^2 - r^{-1}\partial/\partial r$. This has the solution

$$\psi'(x,r,t) = \phi(r)\exp i(\alpha x - \omega t), \quad (2.3)$$

provided ϕ satisfies the inviscid axisymmetric Orr–Sommerfeld (or Rayleigh) equation

$$\left.\begin{aligned} &\mathscr{L}(\alpha,\omega)\phi = 0, \\ &\mathscr{L} \equiv (\overline{U}(r) - \omega/\alpha)(\mathscr{D}^2 - \alpha^2) - (\mathscr{D}^2\overline{U}). \end{aligned}\right\} \quad (2.4)$$

Now our form (2.30) for $\overline{U}$ gives $\overline{U}/U_0 = 1 + O(e^{-1/r})$ as $r \to 0$, and hence, near $r = 0$,

$$(\mathscr{D}^2 - \alpha^2)\phi = 0,$$

with a conveniently normalized solution, finite at $r = 0$,

$$\phi = 2rI_0'(\alpha r) \sim \alpha r^2 \quad \text{as} \quad r \to 0. \quad (2.5)$$

Likewise, if $\overline{U}$ vanishes rapidly at infinity we have

$$\phi \sim CrK_0'(\alpha r) \quad (2.6)$$

as $r \to \infty$, for $\operatorname{Re}\alpha > 0$, I and K being the usual symbols for the modified Bessel functions of the first and second kinds. The constant C is generally arbitrary but has a definite, though unknown, value for a given normalization such as (2.5). Conditions (2.5) and (2.6) and the differential equation (2.4) define the spectrum of spatial eigenvalues $\alpha(\omega)$ for each real frequency ω.

The adjoint eigenfunction $\tilde{\phi}$ and the operator $\tilde{\mathscr{L}}$ adjoint to $\mathscr{L}$ are defined by

$$\tilde{\mathscr{L}}\tilde{\phi} \equiv \left(\overline{U}(r) - \frac{\omega}{\alpha}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \alpha^2 - \frac{1}{r^2}\right)\tilde{\phi} + 2\frac{d\overline{U}}{dr}\left(\frac{\partial\tilde{\phi}}{\partial r} + \frac{1}{r}\tilde{\phi}\right) = 0, \quad (2.7)$$

with the requirement that $\tilde{\phi}$ be finite at $r = 0$ and small as $r \to \infty$, these implying (again with a convenient normalization)

$$\tilde{\phi} \sim \begin{cases} 2I_1(\alpha r) \sim \alpha r & \text{as} \quad r \to 0, \\ \tilde{C}K_1(\alpha r) & \text{as} \quad r \to \infty, \end{cases} \tag{2.8}$$

for some definite $\tilde{C}$. Now let χ and $\tilde{\chi}$ be any two functions which behave in the same way as the eigenfunctions ϕ and $\tilde{\phi}$ (respectively), both as $r \to 0$ and as $r \to \infty$. Then it is easy to show that

$$\int_0^\infty \tilde{\chi}\mathscr{L}\chi\, dr = \int^\infty \chi\tilde{\mathscr{L}}\tilde{\chi}\, dr. \tag{2.9}$$

Reverting now to (2.1), suppose that the mean profile is slowly varying with axial distance x, i.e. that

$$\overline{\psi} \equiv \overline{\psi}(r, X) \quad \text{where} \quad X = \epsilon x \tag{2.10}$$

is a slow variable, and ϵ is a small parameter which indicates the slowness of the jet spreading. Introduce a strained fast variable

$$\eta = g(X)/\epsilon, \tag{2.11}$$

where the function g is to be found under the constraint $g(X) = O(X)$ as $X \to 0$. The idea is that the strained co-ordinate η should play the same role in the slowly diverging flow as x does in strictly parallel flow, i.e. the fast-variable dependence should be exponential, $\psi' \sim \exp i\eta$, the remaining space dependence being taken up in the form of 'amplitude functions' of the slow variable X.

With $\overline{\psi} \equiv \overline{\psi}(r, X)$ and $\psi' \equiv \psi'(r, \eta, X, t)$, equation (2.1) takes the form

$$\left(\frac{\partial}{\partial t} + \overline{U}(r, X)\frac{\partial}{\partial x}\right) D^2\psi' - (\mathscr{D}^2\overline{U})\frac{\partial \psi'}{\partial x}$$
$$= -\epsilon\left\{\frac{1}{r}\frac{\partial \psi'}{\partial r}\frac{\partial}{\partial X}\mathscr{D}^2\overline{\psi} - \frac{1}{r}\frac{\partial \overline{\psi}}{\partial X}\frac{\partial}{\partial r}D^2\psi' + \frac{2}{r^2}\frac{\partial \overline{\psi}}{\partial X}D^2\psi'\right\} + O(\epsilon^2). \tag{2.12}$$

Assume an expansion

$$\psi' = \exp(-i\omega t)\{f_0(r, \eta, X) + \epsilon f_1(r, \eta, X) + \ldots\}$$

and try a modal solution

$$f_0(r, \eta, X) = \exp(i\eta) f_0(r, X)$$

at leading order, to get

$$(\overline{U}(r, X) - \omega/g'(X))(\mathscr{D}^2 - g'^2(X)) f_0(r, X) - (\mathscr{D}^2\overline{U}) f_0(r, X) = 0. \tag{2.13}$$

This is the local Orr–Sommerfeld equation for the equivalent parallel flow problem involving the mean velocity profile at station X. Let $\alpha(X)$ and $\phi(r, X)$ denote the eigenvalue wavenumber and eigenfunction [with a *definite normalization*, say that of (2.5)]. Then the solution of (2.13) is

$$\left.\begin{aligned} f_0(r, X) &= A(X)\phi(r, X), \\ g(X) &= \int_{X_0}^X \alpha(\xi)\, d\xi, \end{aligned}\right\} \tag{2.14}$$

so that
$$f_0(r,\eta,X) = A(X)\,\phi(r,X)\exp\left\{i\int_{X_0}^{X}\frac{\alpha(\xi)\,d\xi}{\epsilon} - i\omega t\right\} \tag{2.15}$$
$$= A(X)\,\phi(r,X)\exp\left\{i\int_{x_0}^{x}\alpha(\epsilon\xi')\,d\xi' - i\omega t\right\}. \tag{2.16}$$

The expression (2.16) displays a typical WKB form, in which the usual fast-variable behaviour $\exp(i\alpha x)$ is uniformized to
$$\exp\left(i\int^{x}\alpha\,dx\right),$$
while the slow variation is taken up in the 'amplitude' $A(X)$ and in the eigenfunction $\phi(r,X)$. Note, however, that amplitude and phase information is contained in *each* of the three terms in (2.15), and is transferred from one term to another by different choice of the normalization imposed on ϕ. In particular, therefore, the effective growth rate and phase speed of a disturbance at any given station X are not identical with the growth rate and phase speed of the disturbance with respect to the local parallel flow at X. Further, the effective growth rate and phase speed are different for different choices of flow variable (pressure, velocity, energy density, etc.), and even for a given variable depend upon the cross-stream location at which they are evaluated. Differences of this kind have been observed in experiments (Chan 1974*a*), and will be discussed again later.

An equation for $A(X)$ follows, in the usual multiple-scales manner, from consideration of (2.12) at $O(\epsilon)$, which gives the following inhomogeneous equation for f_1:
$$\left(-i\omega+\overline{U}\alpha\frac{\partial}{\partial\eta}\right)\left(\mathscr{D}^2+\alpha^2\frac{\partial^2}{\partial\eta^2}\right)f_1-(\mathscr{D}^2\overline{U})\,\alpha\frac{\partial f_1}{\partial\eta}$$
$$= \exp i\eta\left\{\mathscr{F}_{\partial/\partial r}\frac{\partial f_0(r,X)}{\partial X}+\mathscr{G}_{\partial/\partial r}\,f_0(r,X)\right\}, \tag{2.17}$$
where
$$\mathscr{F} \equiv (\overline{U}\alpha-\omega)\,2\alpha-\overline{U}(\mathscr{D}^2-\alpha^2)+(\mathscr{D}^2\overline{U}),$$
$$\mathscr{G} \equiv (3\overline{U}\alpha-\omega)\frac{d\alpha}{dX}-\frac{1}{r}\left(\frac{\partial}{\partial X}\mathscr{D}^2\overline{\Psi}\right)\frac{\partial}{\partial r}+\frac{\partial\overline{\Psi}}{\partial X}r\frac{\partial}{\partial r}\left[\frac{1}{r^2}(\mathscr{D}^2-\alpha^2)\right]. \tag{2.18}$$

Since the terms on the right of (2.17) contain the Orr–Sommerfeld eigensolution itself, we must expect f_1 to contain the familiar kind of secular terms, and so we anticipate a solution of the form
$$f_1(r,\eta,X) = \gamma\eta\exp(i\eta)f_0(r,X)+\exp(i\eta)\,h(r,X). \tag{2.19}$$
This leads to an inhomogeneous Orr–Sommerfeld problem for $h(r,X)$;
$$i\alpha\mathscr{L}(\alpha(X),\omega)\,h(r,X) = \mathscr{F}\,\partial f_0/\partial X+\mathscr{G}f_0+\gamma\mathscr{H}f_0, \tag{2.20}$$
with
$$\mathscr{H} \equiv -2\omega\alpha^2+(\mathscr{D}^2\overline{U})\,\alpha-\overline{U}\alpha\mathscr{D}^2+3\overline{U}\alpha^3,$$
and (2.20) has a solution for h which is finite at $r=0$ and vanishes as $r\to\infty$ if and only if
$$\int_0^\infty \tilde{\phi}(r,X)\left(\mathscr{F}\frac{\partial f_0}{\partial X}+\mathscr{G}f_0+\gamma\mathscr{H}f_0\right)dr = 0, \tag{2.21}$$

where $\tilde{\phi}(r, X)$ is the adjoint eigensolution for the local parallel flow at station X. Provided

$$\int_0^\infty \tilde{\phi}\mathscr{H}f_0\,dr \neq 0$$

this determines γ uniquely, and leads through (2.19) to a correction function ϵf_1 which is $O(\epsilon\eta)$ relative to f_0. This, however, is unacceptable, but not for the usual reason that it is a secular correction which becomes unbounded for sufficiently large η, for η does not necessarily become large for any value of X. The reason for rejecting a term $O(\epsilon\eta)$ is that $\epsilon\eta$ is in fact $g(X)$, and therefore the correction is generally $O(1)$ rather than $o(1)$ once X is as large as $O(1)$. The required vanishing of γ then leads to the equation (cf. Bouthier 1972)

$$m(X)\,dA/dX + n(X)\,A = 0, \tag{2.22}$$

where
$$m(X) = \int_0^\infty \tilde{\phi}(r, X)\mathscr{F}_{\partial/\partial r}\phi(r, X)\,dr, \tag{2.23a}$$

$$n(X) = \int_0^\infty \tilde{\phi}(r, X)\left\{\mathscr{F}_{\partial/\partial r}\frac{\partial\phi}{\partial X}(r, X) + \mathscr{G}_{\partial/\partial r}\phi(r, X)\right\}dr. \tag{2.23b}$$

Thus the stream function is determined to $O(1)$ by the expression, supposedly uniformly valid in x,

$$\psi' = A_0\phi(r, X)\exp\left\{i\int_{X_0}^{X}\frac{\alpha(\xi)\,d\xi}{\epsilon} - \int_{X_0}^{X}\frac{n(\xi)\,d\xi}{m(\xi)}\right\}. \tag{2.24}$$

A similar expression for the pressure perturbation in the wave mode may be obtained directly from the momentum equation, namely

$$p' = A_0P(r, X)\exp\left\{i\int_{X_0}^{X}\frac{\alpha(\xi)\,d\xi}{\epsilon} - \int_{X_0}^{X}\frac{n(\xi)\,d\xi}{m(\xi)}\right\}, \tag{2.25}$$

where
$$P(r, X) = \left(\frac{\omega}{\alpha(X)} - \bar{U}(r, X)\right)\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right) + \frac{\partial\bar{U}}{\partial r}(r, X)\frac{\phi}{r}. \tag{2.26}$$

The parameter ϵ has now served its purpose in enabling fast and slow variations to be formally identified, but in this inviscid problem it is somewhat artificial. We shall therefore dispense with the distinction between x and X, writing $A(x), m(x)$, etc., for the quantities hitherto written as $A(X)$, $m(X)$, etc. Then the $O(1)$ solution for the stream function is

$$\psi' = A_0\phi(r, x)\exp\left\{i\int_{x_0}^{x}\alpha(x')\,dx' - \int_{x_0}^{x}\frac{n(x')}{m(x')}dx'\right\}. \tag{2.27}$$

From this solution a local wavenumber $\bar{\alpha}$ can be defined for any flow variable Q (more precisely for any Q which is a linear functional of ψ'), for any axial station x and for any radial location r (and, of course, for each real frequency ω) by

$$\bar{\alpha}(x, r \mid Q) = -i\frac{\partial}{\partial x}\ln Q(r, x). \tag{2.28}$$

This wavenumber has real and imaginary parts $\bar{\alpha}_R$ and $\bar{\alpha}_I$, which may be interpreted in terms of the local phase speed (or wavelength) and local spatial growth rate for the quantity Q in the neighbourhood of (x, r), and the notation emphasizes the fact that $\bar{\alpha}$ is not just a function of x, but also of r and of the flow quantity under consideration. Thus, for example, the value of $\bar{\alpha}$ for the axial velocity component u_x is

$$\bar{\alpha}(x, r \mid u_x) = -i\frac{\partial}{\partial x}\ln\frac{\partial \phi}{\partial r}(r, x) + \alpha(x) + i\frac{n(x)}{m(x)}, \tag{2.29}$$

in which $\alpha(x)$ is the contribution from the local parallel flow at x, $+in(x)/m(x)$ is a function of x alone, present in all linear functionals of ψ', while the first term depends on r as well as x, and varies with the choice of flow variable. This makes it very clear that mere consideration of the variation of α with x is not necessarily of any value whatever, if one is really interested in the correspondence between theory and *measured* wavelengths and growth rates [the measured quantities always corresponding to those in (2.29)]. For all that is guaranteed is that $\bar{\alpha}$ and α both tend to the same constant α_0 as $\epsilon \to 0$, and this does not preclude just as much of the difference between $\bar{\alpha}$ and α_0 from arising out of the first and last terms of (2.29) as out of the local wavenumber $\alpha(x)$. And then again, just how much comes from the first or last term in (2.29) depends *entirely* on the normalization adopted for ϕ. Recent work by Chan (1974*a*) is open to criticism on these grounds; his §D and figures 10, 15, 16 and 17 all compare experimental results with theoretical predictions involving only the $\alpha(x)$ term in (2.29) without recognizing the possible importance of the other terms in (2.29). Although this invalidates the theoretical side of the work, Chan's paper is nonetheless valuable in providing clear experimental evidence for the variation of growth rate with the flow quantity and with radial location (though his theory (1974*a*) of course precludes both of these possibilities).

In the next section we present computed values for various quantities derived from (2.24) and (2.25). Some limited comparison will be made with the experimental data of Crow & Champagne (1971) and Moore (1977). In most other published data there are various unknowns (such as distance from the nozzle, for example, in many of the figures of Chan 1974*a*) which make further comparison difficult, while there is often in addition the contaminating effect of nonlinearity, mentioned later.

We use a form for the mean velocity which has been used to good effect by Michalke (1971) in the parallel flow approximation. Michalke takes

$$\bar{U}(r) = \frac{U_0}{2}\left\{1 + \tanh\left[b\left(\frac{R}{r} - \frac{r}{R}\right)\right]\right\}, \tag{2.30}$$

where the parameter $b = D/8\delta$, δ is a momentum thickness, R is the jet radius, defined by $\bar{U}(R) = \frac{1}{2}U_0$, and $D = 2R$ is the jet diameter. According to Michalke this form gives reasonable agreement with the profile measured by Crow & Champagne (1971) around the station $x = 2D$, where exponential growth appeared to occur, provided b takes the value $\frac{25}{16}$. The form (2.30) can be simply and appropriately generalized to describe the mean axial velocity throughout, say, the first six diameters from the nozzle. For, in that region, well-known similarity rules (see, for example, Tennekes & Lumley 1972, p. 134) state that velocities are invariant with x, while the turbulence length scales increase linearly with x. Thus we take

$$\overline{U}(r,x) = \frac{U_0}{2}\left\{1+\tanh\left[\frac{D}{8\gamma(x+x_0)}\left(\frac{R}{r}-\frac{r}{R}\right)\right]\right\} \tag{2.31}$$

for any divergence rate γ and virtual origin x_0 such that

$$1+x_0/2D \approx 1/25\gamma \tag{2.32}$$

in order to maintain consistency with Michalke at $x = 2D$. This form simulates the plug flow in the potential core very well, since $\overline{U} \rightarrow U_0$ exponentially as $r \rightarrow 0$. From (2.31) the mean stream function and its derivatives are found as required by numerical operations.

All of the calculations presented in §3 refer to the mean velocity profile for which

$$\delta = \tfrac{3}{100}(x+\tfrac{4}{3}D) \tag{2.33}$$

though we also considered the effect of slight variations in the divergence rate and virtual origin consistent with an equally acceptable representation of mean profiles measured by Crow & Champagne and by Moore. For such point quantities as the phase speed or growth rate these slight variations had only a slight effect; the phase speed seemed particularly insensitive to them. Integrated quantities like the amplitude gain varied considerably from one profile to another if the gain was reckoned in each case relative to a fixed axial location, $x/D = 1$ say. These variations were much reduced if the origin of the gain was taken a fixed distance from the virtual origin of each family of velocity profiles. Even then, however, the remaining variation, arising from differences in spreading rate between the profiles, was considerably larger in the case of the integrated gain functions than in the case of point functions.

We confine attention, somewhat arbitrarily, to the region $x \geqslant D$. For smaller values of x (and arguably even for larger values of x up to perhaps a wavelength of the instability mode) the spatial inhomogeneity caused by the presence of the jet-pipe must be dominant (cf. Orszag & Crow 1970; Crighton 1972). This is borne out by many of the figures given by Crow & Champagne, which show a rapidly distorted fluctuation profile in the first diameter, followed by a nearly exponential growth between $x/D = 1$ and $x/D = 3$, say, beyond which that growth is quickly arrested by the divergence of the mixing layer. We hope in a future paper to return to the issue of the tailpipe effect. This is a strongly frequency-dependent effect which must greatly influence the total gain (relative to the nozzle rather than to $x/D = 1$) experienced by a wave; were it not for the inhibiting effect of the tailpipe a high frequency mode would suffer enormous gains as it was rapidly amplified on the very thin shear layer close to the nozzle.

Finally we question whether the 'slowly varying' approximation is likely to be valid here. If the momentum thickness is regarded as the significant dynamical variable then the rate of spread of the jet in relation to a disturbance of wavelength $O(D)$ is characterized by

$$\epsilon \sim \frac{d}{d(x/D)}\left(\frac{\delta}{D}\right) \sim \tfrac{3}{100},$$

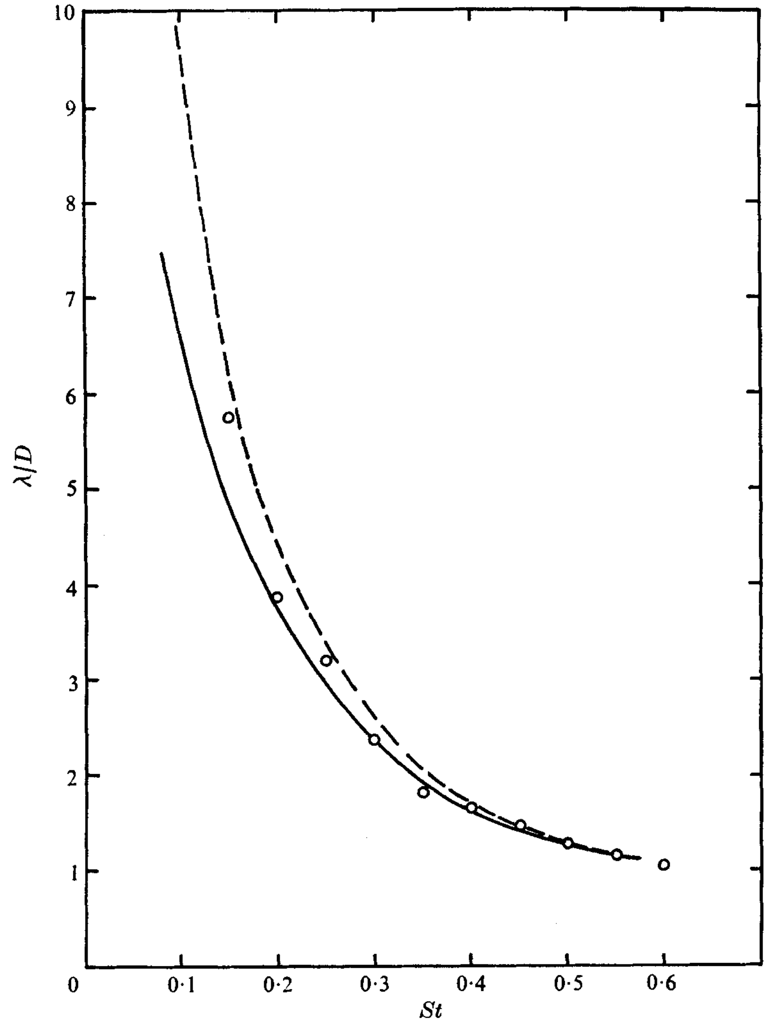

FIGURE 1. Wavelength as a function of Strouhal number. ——, present theory, diverging flow, centre-line axial velocity at $x/D = 2$; – – –, theory for parallel flow with the same mean profile at $x/D = 2$ as above; ○, measurements (centre-line axial velocity) of Crow & Champagne (1971, table 4).

which is perhaps small enough to justify (2.27), though an ϵ of 20–25 % would be indicated by the use of the nominal mixing-layer thickness as the important parameter, that thickness becoming comparable with the diameter D around the end of the potential core, $x \sim 5D$.

3. Computed results and comparison with experiment

Figure 1 shows the calculated variation of wavelength with Strouhal number. The wavelength is taken with reference to the axial velocity fluctuation on the centre-line, and is defined by

$$\lambda = 2\pi/\mathrm{Re}\,\overline{\alpha}(x = 2D, r = 0 \mid u_x). \tag{3.1}$$

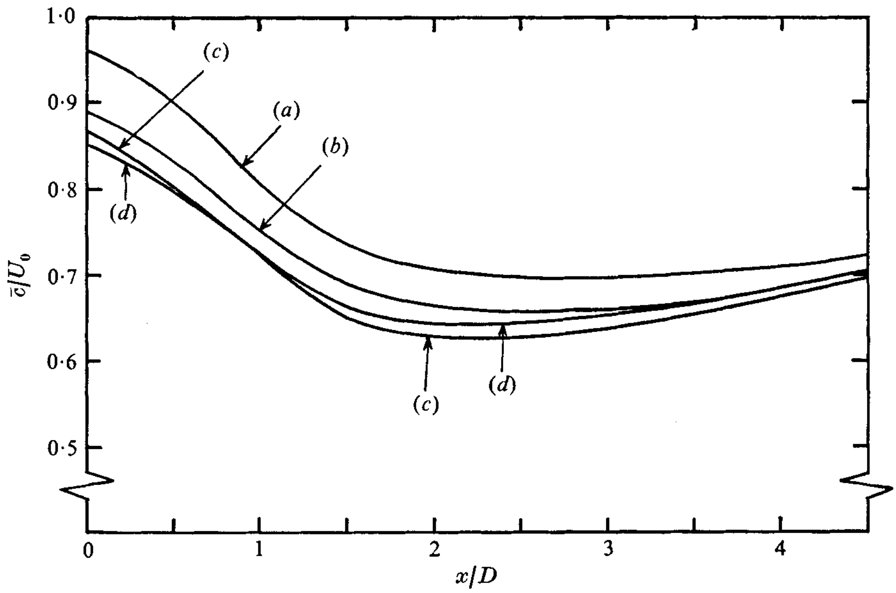

FIGURE 2. Local phase speed $\bar{c}/U_0$ as a function of axial distance x/D. The frequency parameter $\sigma = \omega R/U_0$ has the value 0·9, i.e. $St = 0{\cdot}285$. (*a*) Phase speed for centre-line axial velocity. (*b*) Phase speed for centre-line pressure. (*c*) Phase speed for shear-layer pressure ($r = R$). (*d*) Phase speed for near-field pressure ($r = D$).

The figure shows also the wavelength for a strictly parallel flow with the same profile at $x = 2D$; these values of the wavelength are identical to those calculated by Michalke (1971). It is evident that the results for the strictly parallel flow agree well with those for the diverging flow, especially, as might be expected, at the higher Strouhal numbers. Either set of results agrees remarkably well with the measurements of Crow & Champagne (1971, table 4). This explains Michalke's success in predicting the Crow & Champagne wavelengths from parallel flow theory. As already noted, the wavelength is totally insensitive to small changes in the family of profiles.

Figure 2 shows the variation in the phase speed

$$\bar{c} = \omega/\mathrm{Re}\,\bar{\alpha}(x, r\,|\,p \quad \text{or} \quad u_x) \tag{3.2}$$

with axial position at a fixed value, 0·9, of the frequency parameter $\sigma = \omega R/U_0$. The different curves refer to calculations of the centre-line axial velocity, centre-line pressure, pressure in the middle ($r = R$) of the mixing layer and the pressure at $r = D$ in the near field. Again, all the calculations here refer to (2.33); only small differences arose in the phase speeds for small changes of profile. We have no directly comparable experimental data for this case. However, the trends shown in figure 2 are consistent with figures 4 and 5 of Chan (1974*a*), in that pressure measurements give lower phase speeds than velocity measurements and in that the phase speeds derived from shear-layer and near-field measurements are lower than those on the centre-line. Further, the decrease in $\bar{c}/U_0$ to a minimum of about 0·65 around $x = 2D$ (at this particular value $\sigma = 0{\cdot}9$) is in good general agreement with data given by Moore (1977), though it is not clear

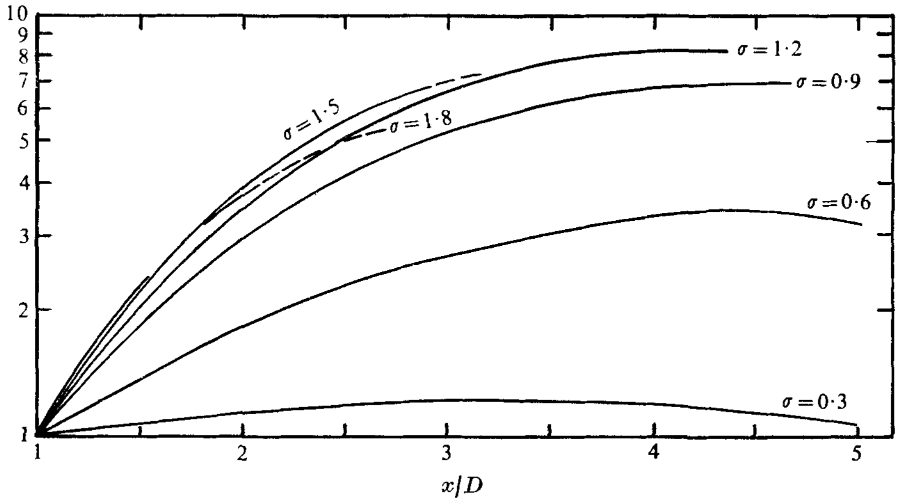

FIGURE 3. Gain in centre-line pressure fluctuation $re\ x/D = 1$. The frequency parameter $\sigma = \omega R/U_0 = \pi St$ has the value shown against each curve.

whether his data in this instance refer to velocity or to pressure measurements. We emphasize here again that any attempt to discuss the variation of $\bar{c}$ with x purely from an examination of the local parallel flow stability problem is *meaningless*.

Figure 3 shows the gain ($re\ x/D = 1$) of the centre-line pressure amplitude with x/D for various values of σ. Michalke (1971) has shown that the effect of a finite momentum thickness is, in parallel flow theory, to single out a most rapidly amplified mode at, say, $x = 2D$. Here we see that the effect of shear-layer divergence is to single out a mode which suffers the greatest total gain ($re\ x/D = 1$). For the profile (2.33) that gain is equivalent to amplification of the pressure amplitude by a factor of about 8, and is attained for $\sigma \approx 1{\cdot}2$ at $x \approx 4D$. Modes with different values of σ are also predicted to achieve a finite total gain, that being achieved at smaller x/D the higher the value of σ. It is plausible that, if the effect of the nozzle and tailpipe were included and the gain reckoned relative to the nozzle exit, this would discriminate against both higher and lower frequencies and still lead to a preferred value of σ between 0·9 and 1·5. In any case it is clear that the emergence of a 'preferred mode' achieving the greatest total gain is the outcome not just of the linear parallel flow amplification considered by Michalke but also of the effects caused by flow divergence and the jet-pipe (and also, no doubt, nonlinearity at the higher forcing levels). Although one might expect eddy damping to be an important mechanism for limiting the total gain, it does not appear from figure 3 that that is actually the case. The net gain by a factor of 8 relative to $x/D = 1$ is very close to that measured by Moore (1977), as is the location ($x = 3\frac{1}{2}D$–$4D$) at which the pressure maximum is attained. On the other hand the minimum Mach number in Moore's experiments was 0·3, and the effect of compressibility on the gain is not known, so that all we can definitely conclude is that the combined effects of compressibility and eddy damping are small.

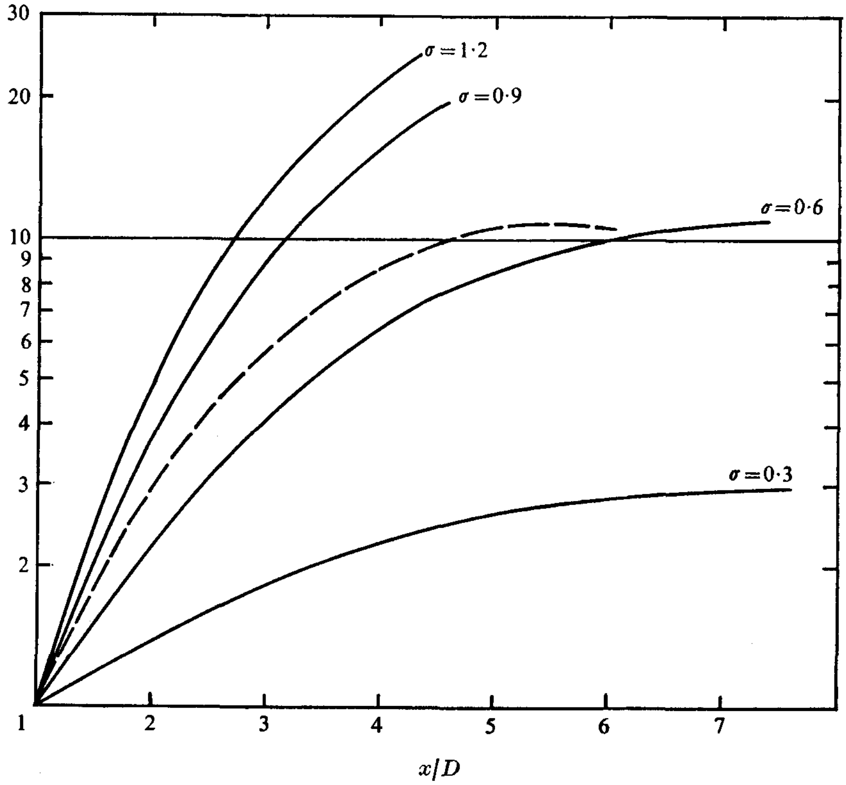

FIGURE 4. Gain in centre-line axial velocity fluctuation *re* $x/D = 1$. ——, calculated values; – – –, measurements of Crow & Champagne (1971, figure 24). The frequency parameter $\sigma = \omega R/U_0 = \pi St$ has the value shown against each curve.

Figure 4 shows the gain curves for the centre-line axial velocity fluctuation. The velocity gain greatly exceeds the pressure gain, by a factor depending on the value of σ and on details of the profile, but typically around 4, and the peak gain for the pressure is reached much earlier than the peak in velocity. Again, these trends are fully consistent with Moore's results, though we encountered difficulties in the computation which made it impossible to follow the velocity to peak amplitude, and we therefore can give no reliable estimate of the total velocity gain for values of σ greater than 0·6. The velocity gain measured by Crow & Champagne (1971) is also shown on figure 4. As noted before, Moore shows that none of their forcing levels was low enough to ensure linearity of the jet response, so it is not surprising that our prediction is well in excess of their measurements, and much closer to those of Moore.

It was shown above that the phase speed or wavelength could be adequately calculated at any station from the locally parallel flow. That is precisely true, within the 'slowly varying' approximation, of the ratio of pressure to axial velocity on the centre-line, which according to (2.24)–(2.26) is given by

$$\left(\frac{p'}{u_x}\right)(x,0,\omega) = \left(\frac{\omega}{\alpha(x)} - U_0\right) \tag{3.3}$$

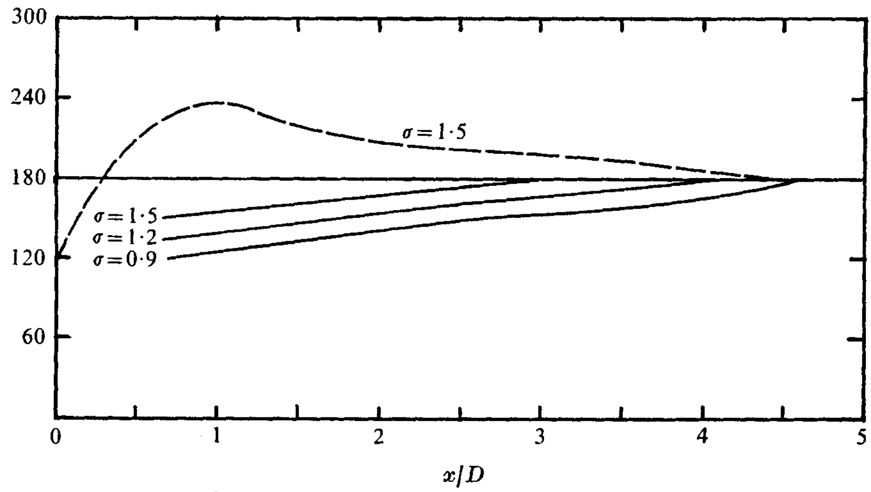

FIGURE 5. Phase angle in degrees by which the pressure fluctuation leads the axial velocity fluctuation on the centre-line. ——, calculated values; – – –, measurements of Moore (1976). The frequency parameter $\sigma = \omega R/U_0$ has the values indicated.

and so is determined by $\alpha(x)$ alone. In particular the phase angle by which the pressure leads the velocity is found to vary with x in the manner shown in figure 5, and is there compared with direct measurements of this angle by Moore (1977) for $\sigma = 1{\cdot}5$. Calculated and measured values both tend to 180° with increasing x/D, but the approach is from different directions. Moore also noted that this unsatisfactory behaviour follows from calculations based on Chan's work (1974*a*–*c*). It may be appropriate, however, to regard the discrepancy as a measure of the importance of eddy damping.

4. Discussion

The aim of this work has been to incorporate the effect of shear-layer divergence into jet stability theory in a simple but rational manner. Integral formulations offer the possibility of dealing with more rapid growth than can be treated by multiple-scales expansions; they also offer the possibility of modelling the processes of energy transfer between the mean flow, the instability wave and the fine-scale turbulence. A number of specific assumptions need, however, to be made before the integral energy method leads to a closed set of equations, and it becomes difficult to identify general mechanisms and trends on the basis of one particular set of assumptions. We have, accordingly, restricted ourselves to one issue, and have shown that simple allowance for slow axial development of the mean jet profile leads (with the exception of the phase-angle variation of figure 5) at worst to general trends in agreement with those found in several independent sets of measurements and at best to excellent numerical agreement for the wavelength, phase speed and pressure gain. Further, these calculations show clearly that the figures apparently regarded as universal by Crow & Champagne are in some cases (the gain, for example) far from universal, and in fact sensitive to profile details, and that significant differences may exist – even for the same profile – between quantities derived from pressure and velocity measurements (and depending, to a lesser extent, on the transverse location of measurement). The fact just is that the modes on a diverging jet are appreciably distorted, and parallel flow ideas and terminology must not be carried over too loosely to the interpretation of actual measurements.

The calculations point to the need for more work in several areas. First, they draw attention to the fact that the tailpipe has an important effect in the early stages, and this is a frequency-dependent effect which affects the selection of a preferred mode and the net gain relative to the nozzle exit. Second, the predicted gains are, as expected, a little too large and are attained a little too far downstream, though in dB terms the errors are not great and in all respects the qualitative behaviour is correctly reproduced here. Hopefully the difference might be made up by the turbulent eddy viscosity, as described by Crow (1968). Third, Michalke's (1971) calculations indicate that spiral modes with azimuthal wavenumber $n = \pm 1$ should be amplified at about the same rate as the axisymmetric modes on a parallel flow profile like that of (2.30), and, moreover, that on parallel flow theory the modes with $n = \pm 1$ should continue to be amplified on the bell-shaped sort of profile which is found in the fully developed jet ($x \gtrsim 8D$), whereas there the axisymmetric modes should only decay. These indications now have experimental support from Moore (1977) and Chan & Templin (1974), so that there is a need to extend the present work to the $n = \pm 1$ modes, for which the governing equations cannot be written as a single equation for a stream function.

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